Self Consistent Random Phase Approximation within the O(5) model and Fermi transitions

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Abstract

Self Consistent Quasiparticle Random Phase Approximation (SCQRPA) is considered in application to the Fermi transitions within the O(5) model. It is demonstrated that SCQRPA improves on renormalized QRPA (RQRPA), a method that has recently become rather popular in this context. The analytical form of the SCQRPA vacuum is used to evaluate all the matrix elements. The SCQRPA results show a general trend similar to the exact solutions. The necessity to change the single particle basis beyond the transition point, and to include the proton-proton and neutron-neutron channels in the QRPA operator, in addition to the proton-neutron one, is pointed out.

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1 Introduction

Since Vogel and Zirnbauer [1] and Cha [2] have discovered the importance of the particle-particle force in the $S = 1$, $T = 0$ channel, the quasiparticle random phase approximation (QRPA) has been the most frequently used nuclear structure method for evaluating double beta ($\beta\beta$) rates both for the two-neutrino decay mode ($\beta\beta_{2\nu}$) and for the neutrinoless mode ($\beta\beta_{0\nu}$). The general feature of this method is that the resulting nuclear matrix elements $\mathcal{M}_{2\nu}$ and $\mathcal{M}_{0\nu}$ turn out to be highly sensitive to the particle-particle force [1, 3, 4, 5, 6, 7, 8]. Moreover, very close to its physical value the QRPA collapses. One thus may suspect that this method yields relatively small values of $\mathcal{M}_{2\nu}$ simply because the approximation breaks down. In other words, the smallness of $\mathcal{M}_{2\nu}$ in the QRPA could be just an artifact of the model.

In the past few years, several modifications of the QRPA have been proposed to improve on the above mentioned behaviour [9, 10, 11, 12, 13]. Yet, the one that has received major attention lately is the so-called renormalized QRPA (RQRPA) [14, 15, 16, 17, 18, 19, 20, 21]. The new ingredient that is brought up is the effect of the ground state correlations (GSC) in the QRPA equations of motion (EOM) i.e. while in the standard QRPA the correlated ground state is approximated by the BCS vacuum, in the RQRPA the GSC are partially taken into account. Let us recall that the GSC are fully incorporated only in the self-consistent RPA (SCRPA) theory [22, 23], which simultaneously leads to a coupling of the single-particle field to the RPA excitations. The same formalism was also generalized to the superconducting systems [24], and when applied in this context will be referred to as the self-consistent QRPA (SCQRPA).

The RQRPA generally yields better results than the QRPA, in the sense that the instability is avoided, and the matrix elements $\mathcal{M}_{2\nu}$ and $\mathcal{M}_{0\nu}$ are somewhat less sensitive to the particle-particle interaction. Yet, the charge-exchange sum rules [25], both for Fermi (F) and Gamow-Teller (GT) transitions, are significantly violated within the RQRPA [15, 17, 18]. It has been shown [17] that the origin of this violation is the omission of the contribution of the so-called scattering (anomalous) terms to these sum rules. A possible way to solve this problem has been suggested in ref. [24], where the inclusion of these terms in the definition of the excitation operators is proposed.

To explore the effect of the GSC on the $\beta\beta$ matrix elements, several applications of the RQRPA have recently been performed in the framework of the exactly solvable O(5) model [26, 27], which allows for a direct comparison between the exact and approximate results. A similar exercise has also been done within the SCQRPA [28], but by solving the corresponding EOM in an approximate way. For that reason, and in contrast with experience with the SCQRPA approach [22, 23, 29], the excitations energies and matrix elements for the $\beta\beta$ decay were not well reproduced.

In this paper we resume the study performed in ref. [28], but avoiding the approxima-

tions done there. With the aim to clarify the physical meaning of the model parameters, a brief introduction of the O(5) model is presented in Section 2. The full SCQRPA equations for the O(5) model, with the simplifying condition of decoupling the charge-exchage mode from the conserving ones, are derived in Section 3. These equations are solved self-consistently with the generalized BCS equations, which implies the coupling of the quasi-particle mean-field to the charge-exchage excitation. The task has been facilitated by the circumstance that it was possible to construct explicitly the correlated ground state wave function for the schematic O(5) Hamiltonian. In Section 4 the RRPA is formulated as the limit of the SCQRPA when the ground state two-body density is approximated by products of one-body densities. Numerical results for the Fermi transitions and the corresponding $\beta\beta$ decay matrix elements, obtained within different approximations, are compared with each other and confronted with exact results in Section 5. Summarizing conclusions are given in Section 6. Finally, some details of the calculations are displayed in the appendices.

2 The Model

In the model, protons and neutrons occupy only one level and the Hamiltonian

$$\begin{aligned} H &= \epsilon_p N_{pp} + \epsilon_n N_{nn} - \frac{G_p}{4} P^\dagger_{pp} P_{pp} - \frac{G_n}{4} P^\dagger_{nn} P_{nn} \\ &+ \frac{1}{4}\left[F_{pn}\left(N_{pn}N_{np} + N_{np}N_{pn}\right) - G_{pn}\left(P^\dagger_{pn}P_{pn} + P^\dagger_{np}P_{np}\right)\right], \end{aligned} \tag{2.1}$$

is such that it can be written in terms of generators of O(5) [28]

$$\begin{aligned} N_{ab} &= \sum_m c^\dagger_{am} c_{bm}, \\ P^\dagger_{ab} &= P^\dagger_{ba} = \sum_m (-1)^{j+m} c^\dagger_{am} c^\dagger_{b-m}, \\ P_{ab} &= \left(P^\dagger_{ab}\right)^\dagger, \end{aligned} \tag{2.2}$$

where ab stands for pp, nn, pn and np.

To see the physical meaning of the model parameters, we rewrite the model Hamiltonian (2.1) in the coupled basis

$$C^\dagger_{ab;JM} = \left(c^\dagger_a \otimes c^\dagger_b\right)_{JM}, \tag{2.3}$$

i.e.

$$H = \sum_{a=p,n}\left(\epsilon_p + \frac{F_{pn}}{4}\right) N_{aa} - \frac{\Omega}{2}\sum_{a=p,n} G_a C^\dagger_{aa;00} C_{aa;00}$$

$$- \quad \Omega G_{pn} C^{\dagger}_{pn;00} C_{pn;00} + \frac{F_{pn}}{2} \sum_{JM} (-1)^J C^{\dagger}_{pn;JM} C_{pn;JM} \tag{2.4}$$

and compare it with the standard one-level shell model Hamiltonian:

$$\begin{aligned} H^{\rm SM} &= \sum_{a=p,n} \epsilon^{\rm SM}_a N_{aa} + \frac{1}{4} \sum_{JM;a=p,n} [1+(-1)^J] \frac{\langle j^2_a; J||v||j^2_a; J\rangle}{\sqrt{2J+1}} C^{\dagger}_{aa;JM} C_{aa;JM} \\ &+ \sum_{JM} \frac{\langle j_p j_n; J||v||j_p j_n; J\rangle + (-1)^J \langle j_p j_n; J||v||j_n j_p; J\rangle}{\sqrt{2J+1}} C^{\dagger}_{pn;JM} C_{pn;JM}. \end{aligned} \tag{2.5}$$

We conclude that the $\epsilon^{\rm SM}_a$ are related with ϵ_a as

$$\epsilon^{\rm SM}_a = \epsilon_a + \frac{F_{pn}}{4}, \tag{2.6}$$

and that the particle-particle matrix elements for the Hamiltonian (2.1) are:

$$G(aaaa; J) \equiv [1+(-1)^J] \frac{\langle j^2_a; J||v||j^2_a; J\rangle}{\sqrt{2J+1}} = -2\Omega G_a \delta_{J0}, \tag{2.7}$$

and

$$\begin{aligned} G(pnpn; J) &\equiv \frac{\langle j_p j_n; J||v||j_p j_n; J\rangle + (-1)^J \langle j_p j_n; J||v||j_n j_p; J\rangle}{\sqrt{2J+1}} \\ &= -\left(\Omega G_{pn} \delta_{J0} - (-1)^J \frac{F_{pn}}{2} \right), \end{aligned} \tag{2.8}$$

with $2\Omega = 2j+1$. Therefore like particles interact only in the $J=0$ channel (pairing term). But, the unlike particles, besides having a pairing term, also interact in $J \neq 0$ channels. In particular, for odd J, the Hamiltonian (2.1) simulates the $T=0$ pairing. Moreover, it is charge dependent, and the limit of charge independence is achieved when $\epsilon_p = \epsilon_n$, $G_p = G_n = G_{pn}$ and $F_{pn} = 0$.

For further use it is convenient to define the particle-hole matrix elements [1]

$$F(aaaa; J) = -(-1)^J G_a \tag{2.9}$$

[1]The particle-hole matrix elements are defined as [33]:

$$F(adcb; J) = -\sum_{J'} (2J'+1) \left\{ \begin{array}{ccc} a & b & J' \\ c & d & J \end{array} \right\} G(abcd; J').$$

$$F(pnpn;J) = \left(\Omega F_{pn}\delta_{J0} - (-1)^J \frac{G_{pn}}{2}\right), \tag{2.10}$$

$$F(ppnn;J) = -\frac{1}{2}\left(F_{pn} + (-1)^J G_{pn}\right). \tag{2.11}$$

It might be worth noting that to make the Hamiltonian (2.1) charge conserving, even when $F_{pn} \neq 0$, one has to do the replacement:

$$H \to H + \frac{F_p}{4} N_{pp} N_{pp} + \frac{F_n}{4} N_{nn} N_{nn}. \tag{2.12}$$

Now

$$\begin{aligned} G(aaaa;J) &= -2G_a\Omega\delta_{J0} + \frac{F_a}{2}(1+(-1)^J) \\ F(aaaa;J) &= -(-1)^J G_a + \frac{F_a}{2}\left(2\Omega\delta_{J0} - 1\right) \end{aligned} \tag{2.13}$$

and the charge independence is fulfilled for $\epsilon_p = \epsilon_n$, $G_p = G_n = G_{pn}$ and $F_p = F_n = F_{pn}$. In this case the following relations are valid:

$$\begin{aligned} G(aaaa;0) &= 2G(pnpn;0), \\ F(aaaa;0) &= F(pnpn;0) + F(ppnn;0). \end{aligned} \tag{2.14}$$

3 The SCQRPA Equations

3.1 General Formulation

The "nuclei" to be considered here are usually superfluid both for protons and for neutrons. We therefore make first a Bogoljubov transformation to quasiparticle operators $a^\dagger_{am}$,

$$c^\dagger_{am} = u_a a^\dagger_{am} + v_a (-1)^{j+m} a_{a,-m}, \qquad a = p, n. \tag{3.1.1}$$

The SCQRPA states are constructed by the action of the excitation operators [2]

$$\Gamma^\dagger_\nu = \sum_{\tau=pp,nn,pn} X^\nu_\tau \tilde{\mathcal{A}}^\dagger_\tau - Y^\nu_\tau \tilde{\mathcal{A}}_\tau, \qquad \nu = 1,2,3 \tag{3.1.2}$$

on the SCQRPA ground state $|0\rangle$,

$$|\nu\rangle = \Gamma^\dagger_\nu |0\rangle, \tag{3.1.3}$$

[2] In recent works [23, 24] it was proposed to also include anomalous terms $a^\dagger a$ in the excitation operator. Though this potentially is of great importance we do not consider this extension here (see also the discussion at the end of this paper).

with the vacuum condition

$$\Gamma_\nu|0\rangle = 0. \tag{3.1.4}$$

The operators

$$\tilde{\mathcal{A}}^\dagger_\tau = \frac{\mathcal{A}^\dagger_{ab}}{\sqrt{2\Omega(1+\delta_{ab})}}, \tag{3.1.5}$$

which appear in eq. (3.1.2) create pp, nn and pn quasiparticle pairs with $J = 0$:

$$\mathcal{A}^\dagger_{ab} = \sum_m (-1)^{j+m} a^\dagger_{am} a^\dagger_{b,-m}, \qquad a, b = p, n. \tag{3.1.6}$$

and therefore the ansatz (3.1.2) can be considered as a Bogoljubov transformation between fermion pair operators.

The normalization condition

$$\langle\nu|\nu'\rangle = \langle 0|\left[\Gamma_{\nu'}, \Gamma^\dagger_\nu\right]|0\rangle = \delta_{\nu\nu'}, \tag{3.1.7}$$

gives

$$\sum_{\tau'\tau}\left(X^{\nu'*}_{\tau'} D_{\tau'\tau} X^\nu_\tau - Y^{\nu'*}_{\tau'} D^*_{\tau'\tau} Y^\nu_\tau\right) = \delta_{\nu\nu'}, \tag{3.1.8}$$

where

$$D_{\tau'\tau} = \langle 0|\left[\tilde{\mathcal{A}}_{\tau'}, \tilde{\mathcal{A}}^\dagger_\tau\right]|0\rangle. \tag{3.1.9}$$

The SCQRPA equations have then the form:

$$\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \omega \begin{pmatrix} D & 0 \\ 0 & -D^* \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}, \tag{3.1.10}$$

where

$$\begin{aligned} A_{\tau\tau'} &= \langle 0|\left[\tilde{\mathcal{A}}_\tau, \mathcal{H}, \tilde{\mathcal{A}}^\dagger_{\tau'}\right]|0\rangle = A^*_{\tau'\tau}, \\ B_{\tau\tau'} &= -\langle 0|\left[\tilde{\mathcal{A}}_\tau, \mathcal{H}, \tilde{\mathcal{A}}_{\tau'}\right]|0\rangle = B_{\tau'\tau}, \end{aligned} \tag{3.1.11}$$

are the symmetrised double commutators defined by Rowe [30] with

$$\mathcal{H} = H - \lambda_p N_{pp} - \lambda_n N_{nn}. \tag{3.1.12}$$

The SCRPA equations are still incomplete because the mean field parameters u_a, v_a, and λ_a are so far undetermined. This general problem of SCRPA has been solved very satisfactorily in [24] where it has been shown that the mean field parameters (the single particle basis in a non-superfluid system and/or the amplitudes u_a, v_a in a superfluid

one) can be determined in minimizing the SC(Q)RPA ground state energy with respect to these parameters. This minimization turns out to be equivalent to [24]:

$$\langle 0|[\mathcal{H}, \Gamma^\dagger_\nu]|0\rangle = 0; \qquad \nu = 1, 2, 3, \tag{3.1.13}$$

which with (3.1.4) is a very natural equation from the EOM point of view.

The eq. (3.1.13) is equivalent to

$$\langle 0|[\mathcal{H}, \tilde{\mathcal{A}}^\dagger_\tau]|0\rangle = 0, \tag{3.1.14}$$

which, upon replacing $|0\rangle$ by the mean field ground state, turns out to be identical to the usual BCS equation. [3]

The chemical potentials $\lambda_{n,p}$ are determined from the usual particle number condition, *i.e.* it is imposed that the SCQRPA ground state $|0\rangle$ has, on average, the correct number of protons ($Z = \mathrm{N}_p$) and neutrons ($N = \mathrm{N}_n$)

$$\langle 0|N_{aa}|0\rangle = \mathrm{N}_a. \tag{3.1.15}$$

The set of equations (3.1.4), (3.1.10), (3.1.11) and (3.1.15) have to be solved simultaneously, together with the conditions (3.1.14). This generally makes the problem to be very demanding computationally. Indeed the problem has only been tackled so far within very simple schematic models [24]. Here we are going to work out the SCQRPA equations of the O(5) model, under some simplifying conditions. That is, we assume that the charge-exchange mode,

$$\Gamma^\dagger_{pn} = X_{pn}\frac{\mathcal{A}^\dagger_{pn}}{\sqrt{2\Omega}} - Y_{pn}\frac{\mathcal{A}_{pn}}{\sqrt{2\Omega}}, \tag{3.1.16}$$

decouples from the modes involving like particles pairs,

$$\Gamma^\dagger_\nu = \sum_{a=p,n} X^\nu_{aa}\frac{\mathcal{A}^\dagger_{aa}}{2\sqrt{\Omega}} - Y^\nu_{aa}\frac{\mathcal{A}_{aa}}{2\sqrt{\Omega}}. \tag{3.1.17}$$

Since both the Hamiltonian and the Bogoljubov transformation are charge conserving, the decoupling happens automatically for a SCQRPA vacuum that is a superposition of states with even number of protons and neutrons. This is achieved by taking into account only the correlations due to the *pn* pairs. In other words, the SCQRPA vacuum is determined only by the equation

$$\Gamma_{pn}|0\rangle = 0. \tag{3.1.18}$$

[3] These generalized mean field equations also serve to make the double commutators (3.1.11) symmetric. In the O(5) model the symmetry of the double commutators is verified (see eqs. (3.5.9)).

Within this framework, and since

$$\langle 0|[\mathcal{H}, \mathcal{A}^\dagger_{pn}]|0\rangle = 0 \tag{3.1.19}$$

is trivially satisfied, the eqs. (3.1.15) for the mean field reduce to

$$\langle 0|[\mathcal{H}, \mathcal{A}^\dagger_{pp}]|0\rangle = \langle 0|[\mathcal{H}, \mathcal{A}^\dagger_{nn}]|0\rangle = 0, \tag{3.1.20}$$

and the number condition (3.1.14) has to be obeyed.

In summary, since in this paper we are interested in investigating the Fermi transitions in this schematic model, our equations are: (3.1.16) for the excitation operator, (3.1.18) for the SCQRPA vacuum, and (3.1.14) and (3.1.19) for the mean field.

3.2 Hamiltonian in the quasiparticle basis

The first step in solving the above equations is to rewrite the Hamiltonian (3.1.12) in the quasiparticle basis. Using the Bogoljubov transformation, $\mathcal{H}$ can be put in normal order with respect to the quasiparticle vacuum. Five type of terms occur,

$$\mathcal{H} = \mathcal{H}^{00} + \mathcal{H}^{11} + \mathcal{H}^{20} + \mathcal{H}^{22} + \mathcal{H}^{31} + \mathcal{H}^{40}, \tag{3.2.1}$$

and they can be expressed in terms of the generators of an O(5) algebra $\mathcal{A}^\dagger_{ab}$, $\mathcal{A}_{ab} = \left(\mathcal{A}^\dagger_{ab}\right)^\dagger$ and

$$\mathcal{N}_{ab} = \sum_m a^\dagger_{am} a_{bm}, \tag{3.2.2}$$

analogous to those given by (2.2), but now in the quasiparticle basis. The first term

$$\begin{aligned} \mathcal{H}^{00} &= \sum_{a=p,n} \left\{ \Omega \left[2\left(\epsilon^{(0)}_a - \lambda_a\right) v_a^2 - G_a v_a^4 \right] - G_a \Omega^2 u_a^2 v_a^2 \right\} \\ &- \Omega(G_{pn} + F_{pn}) v_p^2 v_n^2, \end{aligned} \tag{3.2.3}$$

is the BCS ground state energy, and the second one

$$\mathcal{H}^{11} = E_p \mathcal{N}_{pp} + E_n \mathcal{N}_{nn}, \tag{3.2.4}$$

is the one-quasiparticle Hamiltonian, where

$$E_a = (\epsilon^{(s)}_a - \lambda_a)(u_a^2 - v_a^2) + 2\Omega v_a^2 u_a^2 G_a, \tag{3.2.5}$$

are the quasiparticle energies, in the BCS limit. Here

$$\epsilon^{(s)}_a = \epsilon^{SM}_a + \mu_a, \tag{3.2.6}$$

are the single particle energies, corrected by the self-energy terms [17]

$$\begin{aligned}\mu_p &= -G_p v_p^2 - \frac{1}{2}(G_{pn}+F_{pn})v_n^2, \\ \mu_n &= -G_n v_n^2 - \frac{1}{2}(G_{pn}+F_{pn})v_p^2. \end{aligned} \tag{3.2.7}$$

The remaining quantities are

$$\mathcal{H}^{20} = \frac{1}{2}\sum_a \left[2(\epsilon_a^{(s)}-\lambda_a)u_a v_a - \Omega G_a u_a v_a (u_a^2 - v_a^2)\right](\mathcal{A}^\dagger_{aa} + \mathcal{A}_{aa}), \tag{3.2.8}$$

$$\begin{aligned}\mathcal{H}^{40} &= \frac{1}{4}\sum_a G_a u_a^2 v_a^2 (\mathcal{A}^\dagger_{aa}\mathcal{A}^\dagger_{aa} + \mathcal{A}_{aa}\mathcal{A}_{aa}) \\ &+ \frac{1}{2}(G_{pn}+F_{pn})u_n v_n u_p v_p (\mathcal{A}^\dagger_{pn}\mathcal{A}^\dagger_{pn} + \mathcal{A}_{pn}\mathcal{A}_{pn}),\end{aligned} \tag{3.2.9}$$

$$\begin{aligned}\mathcal{H}^{31} &= \frac{1}{2}\sum_a G_a u_a v_a (u_a^2 - v_a^2)(\mathcal{A}^\dagger_{aa}\mathcal{N}_{aa} + \mathcal{N}_{aa}\mathcal{A}_{aa}) \\ &+ \frac{1}{2}(G_{pn}+F_{pn})\Big[u_n v_n (u_p^2 - v_p^2)(\mathcal{A}^\dagger_{pn}\mathcal{N}_{np} + \mathcal{N}_{pn}\mathcal{A}_{pn}) \\ &+ u_p v_p (u_n^2 - v_n^2)(\mathcal{A}^\dagger_{pn}\mathcal{N}_{pn} + \mathcal{N}_{np}\mathcal{A}_{pn})\Big],\end{aligned} \tag{3.2.10}$$

and

$$\begin{aligned}\mathcal{H}^{22} &= -\frac{1}{4}\sum_a G_a (u_a^4 + v_a^4)\mathcal{A}^\dagger_{aa}\mathcal{A}_{aa} - \sum_a G_a u_a^2 v_a^2 (\mathcal{N}^2_{aa} - \mathcal{N}_{aa}) \\ &+ \frac{1}{2}[F_{pn}(u_p^2 v_n^2 + u_n^2 v_p^2) - G_{pn}(u_p^2 u_n^2 + v_p^2 v_n^2)]\mathcal{A}^\dagger_{pn}\mathcal{A}_{pn} \\ &- \frac{1}{2}(G_{pn}+F_{pn})u_p v_p u_n v_n (\mathcal{N}^2_{pn} + \mathcal{N}^2_{np}) \\ &+ \frac{1}{4}[F_{pn}(u_p^2 u_n^2 + v_p^2 v_n^2) - G_{pn}(u_p^2 v_n^2 + u_n^2 v_p^2)] \\ &\times (\mathcal{N}_{pn}\mathcal{N}_{np} + \mathcal{N}_{np}\mathcal{N}_{pn} - \mathcal{N}_{nn} - \mathcal{N}_{pp}).\end{aligned} \tag{3.2.11}$$

3.3 The SCQRPA vacuum

The correlated vacuum is defined by eq. (3.1.18), which is equivalent to

$$(\mathcal{A}_{pn} - z\mathcal{A}^\dagger_{pn})|0\rangle = 0; \quad z = \frac{Y}{X}. \tag{3.3.1}$$

From the vacuum condition it is very difficult to find an explicit expression for the SCQRPA ground state $|0\rangle$ in the general case. Yet, because of the simplicity of the model used here, the eq. (3.3.1) can be solved exactly, and one obtains

$$|0\rangle = \sqrt{N_0}\sum_{l=0}^{\Omega} a_l z^l \left(\mathcal{A}^\dagger_{pn}\right)^{2l}|BCS\rangle = \sqrt{N_0}\sum_{l=0}^{\Omega}\alpha_l z^l |2l\rangle, \tag{3.3.2}$$

with $|BCS\rangle$ being the quasiparticle vacuum and

$$N_0 = \left(\sum_{l=0}^{\Omega}\alpha_l^2 z^{2l}\langle 2l|2l\rangle\right)^{-1}, \tag{3.3.3}$$

$$\alpha_l = (2\Omega)^l \frac{\Omega!}{(2\Omega)!}\frac{(2\Omega-2l))!}{l!(\Omega-l)!}; \qquad \langle k|k\rangle = \frac{k!(2\Omega)!}{(2\Omega-k)!(2\Omega)^k}. \tag{3.3.4}$$

Thus, the SCQRPA vacuum is a superposition of states with equal even number of neutron and proton quasiparticles, and as such a superposition of states with even number of protons and neutrons. To solve the SCQRPA equations we have to evaluate the expectation values of the generators $\mathcal{A}^\dagger_{ab}$, $\mathcal{A}_{ab}$ and $\mathcal{N}_{ab}$ and their bilinear combinations as well. The results are given in the Appendix A.

3.4 Generalized BCS equations

The only undetermined quantities are the u_a's and v_a's of the BCS state and the chemical potentials. They are fixed by the generalized BCS equations (3.1.19) and the number conditions (3.1.14). The first ones yield

$$\begin{aligned}\langle 0|[\mathcal{H}^{20}+\mathcal{H}^{31},\mathcal{A}^\dagger_{pp}]|0\rangle &= 0,\\ \langle 0|[\mathcal{H}^{20}+\mathcal{H}^{31},\mathcal{A}^\dagger_{nn}]|0\rangle &= 0,\end{aligned} \tag{3.4.1}$$

and from the second ones we obtain [4]

$$\mathrm{N}_a = 2\Omega v_a^2 + (u_a^2 - v_a^2)\langle 0|\mathcal{N}_{aa}|0\rangle, \quad a = p, n. \tag{3.4.2}$$

When the SCQRPA vacuum $|0\rangle$ is replaced by the quasiparticle vacuum $|BCS\rangle$, the eqs. (3.4.1) and (3.4.2) reduce to the usual BCS equations

$$\begin{aligned}\langle BCS|[\mathcal{H}^{20},\mathcal{A}^\dagger_{pp}]|BCS\rangle &= \left[2(\epsilon_p^{(s)}-\lambda_p) - G_p\Omega(u_p^2-v_p^2)\right]u_p v_p = 0,\\ \langle BCS|[\mathcal{H}^{20},\mathcal{A}^\dagger_{nn}]|BCS\rangle &= \left[2(\epsilon_n^{(s)}-\lambda_n) - G_n\Omega(u_n^2-v_n^2)\right]u_n v_n = 0,\end{aligned} \tag{3.4.3}$$

[4] This generalized particle number equation has for the first time been incorporated into an extended RPA calculation in refs. [15, 24] though this had not explicitly been mentioned in the latter of the two references.

with

$$v_a^2 = \frac{\mathrm{N}_a}{2\Omega}; \qquad\qquad a = p, n. \tag{3.4.4}$$

The SCQRPA correlations lead to the following generalized BCS equations:

$$2\xi_a u_a v_a + \Delta_a(v_a^2 - u_a^2) = 0; \qquad u_a^2 + v_a^2 = 1, \tag{3.4.5}$$

with the standard solution for $u's$ and $v's$

$$\begin{aligned} v_a^2 &= \frac{1}{2}\left(1 - \frac{\xi_a}{\sqrt{\xi_a^2 + \Delta_a^2}}\right), \\ u_a^2 &= \frac{1}{2}\left(1 + \frac{\xi_a}{\sqrt{\xi_a^2 + \Delta_a^2}}\right), \end{aligned} \tag{3.4.6}$$

where

$$\begin{aligned} \xi_p &= \frac{1}{\Omega}(\epsilon_p^{(s)} - \lambda_p)\langle 0|(\Omega - \mathcal{N}_{pp})|0\rangle \\ &- \frac{1}{4\Omega}\left(F_{pn} + G_{pn}\right)\left(v_n^2 - u_n^2\right)\langle 0|\left(\mathcal{A}_{pn}^\dagger \mathcal{A}_{pn} - \mathcal{N}_{np}\mathcal{N}_{pn}\right)|0\rangle, \end{aligned} \tag{3.4.7}$$

and

$$\begin{aligned} \Delta_p &= \frac{1}{2\Omega} G_p u_p v_p \langle 0|\left[2(\Omega - \mathcal{N}_{pp})^2 - \mathcal{A}_{pp}^\dagger \mathcal{A}_{pp}\right]|0\rangle \\ &- \frac{1}{2\Omega}\left(F_{pn} + G_{pn}\right) u_n v_n \langle 0|\mathcal{A}_{pn}^\dagger \mathcal{A}_{pn}^\dagger|0\rangle, \end{aligned} \tag{3.4.8}$$

are, respectively, the renormalized single particle energy and gap parameter for protons (and similarly for neutrons). One remarks that formally the generalized BCS equations have the same structure as the ordinary ones. However proton and neutron equations are coupled, the pairing interaction is renormalized by the two body densities (screening) and also the single particle energies are coupled back to the correlations.

From the number of particle condition (3.4.2) it follows immediately that the BCS amplitudes (3.4.6) can be expressed as:

$$\begin{aligned} v_a^2 &= \frac{1}{2}\left(1 - \frac{\Omega - \mathrm{N}_a}{\Omega - \langle 0|\mathcal{N}_{aa}|0\rangle}\right), \\ u_a^2 &= \frac{1}{2}\left(1 + \frac{\Omega - \mathrm{N}_a}{\Omega - \langle 0|\mathcal{N}_{aa}|0\rangle}\right) = 1 - v_a^2. \end{aligned} \tag{3.4.9}$$

It should be stressed that these expressions, which generalize (3.4.4) are independent of (3.4.6) and are equivalent to the eq. (3.4.2), which fixes the number of particles. They allow us to express directly the quantities entering the BCS eqs. (3.4.5) in terms of the one-body densities and to solve them.

3.5 The SCQRPA equations

As observed previously, the vacuum state $|0\rangle$ is a superposition of states with even number of protons and neutrons. Therefore, since the Hamiltonian is charge conserving, the SCQRPA eqs. (3.1.10) split into two equations, one for the charge-conserving excitations and the other for the charge-exchange excitations. Being interested only in investigating the Fermi transitions, we will consider only the last ones,

$$\Gamma^\dagger_{pn} = X_{pn}\frac{\mathcal{A}^\dagger_{pn}}{\sqrt{2\Omega}} - Y_{pn}\frac{\mathcal{A}_{pn}}{\sqrt{2\Omega}}, \tag{3.5.1}$$

with

$$|pn\rangle = \Gamma^\dagger_{pn}|0\rangle. \tag{3.5.2}$$

The normalization condition

$$\langle pn|pn\rangle = \langle 0|[\Gamma_{pn}, \Gamma^\dagger_{pn}]|0\rangle = 1, \tag{3.5.3}$$

gives

$$\left(X^2_{pn} - Y^2_{pn}\right)\langle 0|[\mathcal{A}_{pn}, \mathcal{A}^\dagger_{pn}]|0\rangle = 2\Omega. \tag{3.5.4}$$

Next, we introduce the normalized pair creation operators

$$\tilde{\mathcal{A}}^\dagger_{pn} = D^{-1/2}_{pn}\frac{\mathcal{A}^\dagger_{pn}}{2\Omega}, \tag{3.5.5}$$

and the normalized amplitudes

$$\mathcal{X}_{pn} = D^{1/2}_{pn}X_{pn}, \qquad \mathcal{Y}_{pn} = D^{1/2}_{pn}Y_{pn}, \tag{3.5.6}$$

with

$$D_{pn} = \langle 0|\frac{[\mathcal{A}_{pn}, \mathcal{A}^\dagger_{pn}]}{2\Omega}|0\rangle = 1 - \frac{\langle 0|\mathcal{N}_{pp} + \mathcal{N}_{nn}|0\rangle}{2\Omega}, \tag{3.5.7}$$

to write the SCQRPA equations as

$$\begin{pmatrix} \bar{A} & \bar{B} \\ \bar{B}^* & \bar{A}^* \end{pmatrix}\begin{pmatrix} \mathcal{X}_{pn} \\ \mathcal{Y}_{pn} \end{pmatrix} = \omega\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \mathcal{X}_{pn} \\ \mathcal{Y}_{pn} \end{pmatrix}. \tag{3.5.8}$$

Here ω is the SCQRPA excitation energy and

$$\begin{aligned} \bar{A} &= \langle 0|\left[\tilde{\mathcal{A}}_{pn}, [\mathcal{H}, \tilde{\mathcal{A}}^\dagger_{pn}]\right]|0\rangle = \langle 0|\left[[\tilde{\mathcal{A}}_{pn}, \mathcal{H}], \tilde{\mathcal{A}}^\dagger_{pn}\right]|0\rangle, \\ \bar{B} &= -\langle 0|\left[\tilde{\mathcal{A}}_{pn}, [\mathcal{H}, \tilde{\mathcal{A}}_{pn}]\right]|0\rangle. \end{aligned} \tag{3.5.9}$$

Using the expression for $\mathcal{H}$ in the quasiparticle basis, one can evaluate the double commutators in eq. (3.5.7), and obtain

$$\begin{aligned} \bar{A} &= \bar{A}^{11} + \bar{A}^{22} + \bar{A}^{40}, \\ \bar{B} &= \bar{B}^{40} + \bar{B}^{22}, \end{aligned} \tag{3.5.10}$$

with

$$\begin{aligned} \bar{A}^{ij} &= \langle 0| \left[\tilde{\mathcal{A}}_{pn}, [\mathcal{H}^{ij}, \tilde{\mathcal{A}}^{\dagger}_{pn}]\right] |0\rangle \\ \bar{B}^{ij} &= -\langle 0| \left[\tilde{\mathcal{A}}_{pn}, [\mathcal{H}^{ij}, \tilde{\mathcal{A}}_{pn}]\right] |0\rangle. \end{aligned} \tag{3.5.11}$$

The explicit expressions for $\bar{A}^{ij}$ and $\bar{B}^{ij}$ are:

$$\bar{A}^{11} = E_p + E_n, \tag{3.5.12}$$

with E_a given by eq. (3.2.5), and

$$\begin{aligned} \bar{A}^{22} &= \frac{1}{2}(F_{pn}\beta - G_{pn}\alpha) + (F_{pn}\alpha - G_{pn}\beta)\frac{\langle 0| \left(2\Omega - \mathcal{N}\right)^2 |0\rangle}{2\langle 0| \left(2\Omega - \mathcal{N}\right) |0\rangle} \\ &- [4G_p u_p^2 v_p^2 + 4G_n u_n^2 v_n^2 + F_{pn}\alpha - G_{pn}\beta]\frac{\langle 0|\mathcal{A}^{\dagger}_{pn}\mathcal{A}_{pn}|0\rangle}{\langle 0| \left(2\Omega - \mathcal{N}\right) |0\rangle} \\ &+ [G_p(u_p^4 + v_p^4) + G_n(u_n^4 + v_n^4) + F_{pn}\beta - G_{pn}\alpha]\frac{\langle 0|\mathcal{A}^{\dagger}_{pp}\mathcal{A}_{pp} - 2\mathcal{N}_{pn}\mathcal{N}_{np}|0\rangle}{\langle 0| \left(2\Omega - \mathcal{N}\right) |0\rangle} \\ &- 4(G_p u_p^2 v_p^2 + G_n u_n^2 v_n^2)\frac{\langle 0|(2\Omega - \mathcal{N})\mathcal{N}|0\rangle}{\langle 0| \left(2\Omega - \mathcal{N}\right) |0\rangle}, \end{aligned} \tag{3.5.13}$$

$$\bar{A}^{40} = -(G_{pn} + F_{pn})u_p v_p u_n v_n \frac{\langle 0|\mathcal{A}^{\dagger}_{pn}\mathcal{A}^{\dagger}_{pn} + \mathcal{A}_{pn}\mathcal{A}_{pn}|0\rangle}{\langle 0| \left(2\Omega - \mathcal{N}\right) |0\rangle}, \tag{3.5.14}$$

$$\bar{B}^{40} = -(G_{pn} + F_{pn})u_p v_p u_n v_n \left(1 - \frac{\langle 0| \left(2\Omega - \mathcal{N}\right)^2 - 2\mathcal{A}^{\dagger}_{pn}\mathcal{A}_{pn}|0\rangle}{\langle 0| \left(2\Omega - \mathcal{N}\right) |0\rangle}\right), \tag{3.5.15}$$

$$\begin{aligned} \bar{B}^{22} &= -[4G_p u_p^2 v_p^2 + 4G_n u_n^2 v_n^2 + F_{pn}\alpha - G_{pn}\beta]\frac{\langle 0|\mathcal{A}_{pn}\mathcal{A}_{pn}|0\rangle}{\langle 0| \left(2\Omega - \mathcal{N}\right) |0\rangle} \\ &+ [G_p(u_p^4 + v_p^4) + G_n(u_n^4 + v_n^4) + F_{pn}\beta - G_{pn}\alpha]\frac{\langle 0|\mathcal{A}_{pp}\mathcal{A}_{nn}|0\rangle}{\langle 0| \left(2\Omega - \mathcal{N}\right) |0\rangle}, \end{aligned} \tag{3.5.16}$$

where the short notations,

$$\mathcal{N} = \mathcal{N}_{pp} + \mathcal{N}_{nn}, \tag{3.5.17}$$

and

$$\alpha = u_p^2 v_n^2 + u_n^2 v_p^2; \qquad \beta = u_p^2 u_n^2 + v_n^2 v_p^2, \tag{3.5.18}$$

have been used.

It easily can be seen that, when the ground state correlations in the SCQRPA equations are neglected, *i.e.* when the SCQRPA vacuum $|0\rangle$ is replaced by the quasiparticle vacuum $|BCS\rangle$, the usual QRPA equations

$$\begin{aligned} A &= \frac{\Omega}{2}(G_p + G_n) - \beta G(pnpn;0) + \alpha F(pnpn;0), \\ B &= [F(pnpn;0) - G(pnpn;0)] u_n v_n u_p v_p, \end{aligned} \tag{3.5.19}$$

are recovered.

4 Renormalized QRPA equations (RQRPA)

As pointed out before, to solve the SCQRPA (SCRPA) equations exactly is very demanding from a computational point of view, and this has not been done so far in realistic cases. Several authors [14, 15, 16, 17, 18, 19, 20, 31, 32] have suggested different approximations to derive simpler equations to be amenable for the numerical calculations. In essence, the simplifying hypothesis consists in factorizing the two-body correlations in the SCQRPA (SCRPA) equations of motion. For a normal system this implies to approximate the ground state two-body density by products of one-body densities,

$$\langle 0|c_a^\dagger c_b^\dagger c_d c_c|0\rangle \cong \langle 0|c_a^\dagger c_c|0\rangle\langle 0|c_b^\dagger c_d|0\rangle - \langle 0|c_a^\dagger c_d|0\rangle\langle 0|c_b^\dagger c_c|0\rangle. \tag{4.1}$$

The parallel approximation in a superfluid system is:

$$\begin{aligned} \langle 0|a_a^\dagger a_b^\dagger a_d a_c|0\rangle &\cong \langle 0|a_a^\dagger a_c|0\rangle\langle 0|a_b^\dagger a_d|0\rangle - \langle 0|a_a^\dagger a_d|0\rangle\langle 0|a_b^\dagger a_c|0\rangle \\ &+ \langle 0|a_a^\dagger a_b^\dagger|0\rangle\langle 0|a_d a_c|0\rangle, \\ \langle 0|a_a^\dagger a_b^\dagger a_d^\dagger a_c|0\rangle &\cong \langle 0|a_a^\dagger a_c|0\rangle\langle 0|a_b^\dagger a_d^\dagger|0\rangle - \langle 0|a_a^\dagger a_d^\dagger|0\rangle\langle 0|a_b^\dagger a_c|0\rangle \\ &+ \langle 0|a_a^\dagger a_b^\dagger|0\rangle\langle 0|a_d^\dagger a_c|0\rangle, \\ \langle 0|a_a^\dagger a_b^\dagger a_d^\dagger a_c^\dagger|0\rangle &\cong \langle 0|a_a^\dagger a_c^\dagger|0\rangle\langle 0|a_b^\dagger a_d^\dagger|0\rangle - \langle 0|a_a^\dagger a_d^\dagger|0\rangle\langle 0|a_b^\dagger a_c^\dagger|0\rangle \\ &+ \langle 0|a_a^\dagger a_b^\dagger|0\rangle\langle 0|a_d^\dagger a_c^\dagger|0\rangle, \end{aligned} \tag{4.2}$$

and the corresponding hermitian conjugates.

The SCQRPA ground state does not have a fixed number of quasiparticles. Thus its quasiparticle densities have both terms that conserve the number of quasiparticles and terms that violate it. In field theory language, one would say that in eq. (3.5.1) we have approximated the four-point function by a product of two-point functions. Note that

even with this approximation, to calculate the one-body densities we have to evaluate the ground state $|0\rangle$ explicitly, and this is very difficult in practice. Therefore, the one-body density (two-point function) is approximated so that it is not required to determine the excitations and the ground state simultaneously. This class of approximations are known as RQRPA (RRPA). They are valid as long as the effect of the two-body correlations can be neglected, and their main advantage, in comparison with SCQRPA (SCRPA), is the numerical simplicity.

In this paper we derive the RQRPA equations for the O(5) model following the scheme described above. The RQRPA limits for the expectation values are shown in the Appendix B. Within this limit, the generalized BCS equations (3.4.5) reduce to

$$\begin{aligned} & \epsilon_p^{(SM)} - \lambda_p - \frac{G_p\Omega}{2}\left[1 - \frac{\langle 0|\mathcal{N}_{pp}|0\rangle}{\Omega}\right](u_p^2 - v_p^2) \\ - \quad & G_p\left[v_p^2 + (u_p^2 - v_p^2)\frac{\langle 0|\mathcal{N}_{pp}|0\rangle}{2\Omega}\right] \\ - \quad & \frac{1}{2}(F_{pn} + G_{pn})\left[v_n^2 + (u_n^2 - v_n^2)\frac{\langle 0|\mathcal{N}_{nn}|0\rangle}{2\Omega}\right] = 0, \end{aligned} \tag{4.3}$$

and an analogous equation for neutrons.

In the BCS limit the self-energy terms for protons are given by (3.2.7), *i.e.* by $F(pppp;0) = -G_p$ times $\mathrm{N}_p/2\Omega$, plus $F(ppnn;0) = -(F_{pn} + G_{pn})/2$ times $\mathrm{N}_n/2\Omega$. In the RQRPA one has to consider the average number of neutrons and protons in the SCQRPA ground state $|0\rangle$ (see eq. (3.4.2)). Thus, the self-energy terms in the RQRPA have the same interpretation as in the BCS case.

Using the number condition (3.4.2), the eq. (4.3) can be rewritten as

$$\epsilon_p^{(SM)} - \lambda_p - G_p\frac{Z}{2\Omega} - (F_{pn} + G_{pn})\frac{N}{4\Omega} - \frac{G_p\Omega D_{pp}}{2}(u_p^2 - v_p^2) = 0, \tag{4.4}$$

with

$$D_{aa} = 1 - \frac{\langle 0|\mathcal{N}_{aa}|0\rangle}{\Omega}. \tag{4.5}$$

Thus in the RQRPA limit the generalized BCS equations are equal to the usual ones, except for the renormalization of the gap parameter by the factor D_{pp}.

On the other hand, in this limit $\bar{A}$ gets contributions only from the 11 and 22 terms, *i.e.* $\bar{A} = \bar{A}^{11} + \bar{A}^{22}$, and $\bar{B}$ only from the 40 term,*i.e.* $\bar{B} = \bar{B}^{40}$. One obtains

$$\begin{aligned} \bar{A} &= \frac{\Omega}{2}(G_pD_{pp} + G_nD_{nn}) - \beta G(pnpn;0)D_{pn} + \alpha F(pnpn;0)D_{pn}, \\ \bar{B} &= [F(pnpn;0) - G(pnpn;0)]D_{pn}u_nv_nu_pv_p, \end{aligned} \tag{4.6}$$

where α and β are given by (3.5.18) and D_{pn} by (3.5.7). Thus, we see that, compared with the QRPA equations (3.5.16), the matrix elements $F(pnpn;0)$ and $G(pnpn;0)$ are now renormalized by D_{pn}, as already was known previously [14]. Yet, our careful renormalization procedure shows that the BCS quasiparticle energy

$$\varepsilon_a^{BCS} = \frac{\Omega G_a}{2} = -\frac{G(aaaa;0)}{4}, \tag{4.7}$$

is renormalized as

$$\varepsilon_a^R = \varepsilon_a^{BCS} D_{aa}. \tag{4.8}$$

This renormalization is a cooperative effect coming both from $\bar{A}^{11}$ and $\bar{A}^{22}$. In particular the former is a consequence of the coupling of the mean field to excitations, embodied in the generalized BCS equations (4.3). [5]

5 Numerical results and discussion

An exhaustive numerical study of the O(5) model has been performed for different values of N, Z and Ω, as well as for a large set of parameters. For the discussion we have selected the following three samples:

$$\begin{aligned}
&\text{(A)}: \ \Omega = 5, \ N = 8, \ Z = 2, \ G_p = G_n = 0.4 \text{ MeV}, \ F_{pn} = 0.4 \text{ MeV}, \\
&\text{(B)}: \ \Omega = 10, \ N = 14, \ Z = 6, \ G_p = G_n = 0.3 \text{ MeV}, \ F_{pn} = 0.3 \text{ MeV}, \\
&\text{(C)}: \ \Omega = 25, \ N = 30, \ Z = 20, \ G_p = G_n = 0.2 \text{ MeV}, \ F_{pn} = 0.2 \text{ MeV}.
\end{aligned}$$

The strength G_{pn}, or more precisely the ratio,

$$s = \frac{G_{pn}}{G_p}, \tag{5.1}$$

[5] To generalize the result (4.8) to the many level case, where

$$\varepsilon_a^{BCS} = -\frac{1}{4u_a v_a}(2j_a+1)^{-1/2}\sum_b (2j_b+1)^{1/2} u_b v_b G(aabb;0),$$

there are at least two possibilities:

$$\varepsilon_a^{R} = -\frac{D_{aa}}{4u_a v_a}(2j_a+1)^{-1/2}\sum_b (2j_b+1)^{1/2} u_b v_b G(aabb;0),$$

and

$$\varepsilon_a^{R} = -\frac{1}{4u_a v_a}(2j_a+1)^{-1/2}\sum_b (2j_b+1)^{1/2} u_b v_b G(aabb;0) D_{ab},$$

The last one has been obtained in ref. [17] from the self-consistency between the residual interaction and the mean field. See also ref. [20].

is kept as free parameter. The values of the single-particle energies are of no relevance in the context of the present work and we have simple chosen $\epsilon_p = \epsilon_n \equiv 0$. We believe that the last case is as "realistic" as it can be within a schematic one-level model. Namely, we mimic the calculation of a nucleus with $A = 50$, within a rather large configuration space, with the Kisslinger and Sorensen [34] estimate for the pairing coupling constant ($G \cong 25/A$ MeV), and the strength F_{pn} of the order of magnitude of G_p ($F_{pn}/G_p = 1$).

5.1 Solutions of the RPA equation

By solving the equation (3.5.8) we get the frequencies ω and the amplitudes $\mathcal{X}$ and $\mathcal{Y}$. From the last ones we can evaluate the $\beta^{\pm}$ strengths for the transitions from the even-even nucleus (N, Z) to the odd-odd nuclei $(N \pm 1, Z \mp 1)$, *i.e.*

$$S^{\pm} = \sum_l |\langle 0_l | T_{\pm} | 0 \rangle|^2; \tag{5.2}$$

where $T_+ = N_{np}$ and $T_- = N_{pn}$. In an exact calculation the summation in eq. (5.2) goes over all states in the odd-odd nuclei. But, within a QRPA-like calculation done here there is only one intermediate state $|0_{int}\rangle$, and

$$\begin{aligned} \langle 0_{int} | T_- | 0 \rangle &= \sqrt{2\Omega D_{pn}} (\mathcal{X} u_p v_n + \mathcal{Y} v_p u_n), \\ \langle 0_{int} | T_+ | 0 \rangle &= \sqrt{2\Omega D_{pn}} (\mathcal{Y} u_p v_n + \mathcal{X} v_p u_n). \end{aligned} \tag{5.3}$$

It has been pointed out in ref. [26] that the sum rule

$$S^- - S^+ = N - Z, \tag{5.4}$$

is violated in the RQRPA. However, it was shown [15, 17, 28] that this does not happens when both the correct particle number condition (3.4.2) is used and only $J = 0$ ground state pn correlations are included. Under these circumstances $\langle 0|\mathcal{N}_p|0\rangle$ equals $\langle 0|\mathcal{N}_n|0\rangle$ and the above sum rule is fulfilled in RQRPA as well as in SCQRPA.

In figures 1 and 2 are plotted the numerical results for ω and S^-, respectively. We do not show the results for S^+ as they follow from S^- and the sum rule (5.4). As expected, for ω the RQRPA results always fall in between those of the QRPA and SCQRPA. The QRPA collapses at a value of $s = s_{\rm crit}$ that decreases when Ω is increased ($s_{\rm crit} = 2.10,\ 1.41$ and 1.10, for the cases (A), (B) and (C), respectively). On the contrary, in RQRPA and SCQRPA there occurs no collapse and ω approaches zero asymptotically when $s \to \infty$. From the figures we see that for $s < s_{crit}$, all three approximations agree, but near the transition point and beyond they differ. This behaviour is more evident in case (A). The exact results for the transition strengths S^- are also shown in figure 2. As evidenced from

this figure, for $s \lesssim s_{\rm crit}$ the approximate $S^{\pm}$ strengths coincide so well with the exact ones, as well as among themselves, that is hard to distinguish them visually. When s approaches $s_{\rm crit}$ the QRPA values of $S^{\pm}$ increase very fast and go to infinity. The SCQRPA results also reproduce nicely the exact results in the neighborhood of the critical value of s in all three cases. Moreover, in the case A, SCQRPA yields approximately correct $S^{\pm}$ strengths even for values of s very far from the physical region for this parameter ($s \cong 1$).The variations of z, $\langle \mathcal{N} \rangle$ and v_n^2 in the case C, as a function of s, are illustrated in figure 3. We see that: i) the approximate calculations of the number of quasiparticles in the ground state of the even-even nuclei only differ from one another for $s \geq s_{\rm crit}$, and ii) they undergo a sudden change for $s \simeq s_{\rm crit}$, where the QRPA value goes to infinity. The variation of v_n^2 with s in SCRPA and RQRPA is a direct consequence of the particle number condition (3.4.2) and is more pronounced in the latter case.

5.2 Excitation energies of the odd-odd nuclei

While the RPA results for transition strengths $S^{\pm}$ can be directly compared with the corresponding exact values, as well as with the experimental data in a realistic case, this *does not* happens with the pn-QRPA energies ω. In fact, one important question is to establish the relationship between ω and the excitation energies $E^{\pm}$ of the intermediate odd-odd nuclei $(N \pm 1, Z \mp 1)$ from the even-even nucleus (N, Z). This question is particularly relevant for the $\beta\beta$ decay. Actually, in most of the related theoretical studies done in the framework of the QRPA, this problem is circumvented by the use of the experimental ground state energies of the initial, intermediate and final nuclei, and only the relative spacing of the excitation energies is coming from the roots of the QRPA equation. (See, for instance, ref. [18].)

In ref. [6] (see also ref. [17]) it was proposed that the relationship between ω and $E^{\pm}$ is given by

$$E^{\pm} = \omega \mp \lambda_p \pm \lambda_n, \tag{5.5}$$

in full analogy with the simple BCS case [35].

However, from a more fundamental point of view, in the framework of the SCQRPA, it is more natural to define these excitation energies as [24]

$$E^{\pm}_{\rm sc} = \omega \mp \langle 0 | [\Gamma_{np}, [\lambda_p N_p + \lambda_n N_n, \Gamma^{\dagger}_{np}]] | 0 \rangle, \tag{5.6}$$

which leads to

$$E^{\pm}_{\rm sc} = \omega \mp (\mathcal{X}^2 + \mathcal{Y}^2) \sum_a \lambda_a (u_a^2 - v_a^2). \tag{5.7}$$

To test the quality of these two prescriptions we should remember the following properties of the energies $E^{\pm}$:

1) for $\epsilon_p = \epsilon_n$ and $F_{pn} = 0$, E^- should pass through zero at $s = 1$, and
2) when the energy shift $\Delta = \epsilon_p - \epsilon_n$ (which can simulate the Coulomb energy displacement) is introduced, $E^\pm \to E^\pm \mp \Delta$.

Obviously, in the exact calculation both of these conditions are fulfilled. Additionally, from the eq. (3.4.5) and the fact that ω does not change by a global shift of the single particle energies, we see that while the relation (5.5), being linear in $\lambda_p - \lambda_n$, obeys the condition 2), the prescription (5.7) does not. On the other hand, from the results shown in figure 4 for the case (B) with $F_{pn} = 0$, we also conclude that only the first prescription for E^- is consistent with the condition 1). Thus, from now on we will not discuss anymore the energies $E^\pm$ based on the eq. (5.7). Why this last definition of the excitation energies leads in our context to unphysical consequences will be investigated in a future work.

In figure 5 are compared the approximated energies E^- with the exact ones. All three approximations exhibit quite similar behavior for $s \lesssim s_{\rm crit}$. Within this range of values for s, they also agree reasonable well with the exact results. Yet, for $s > s_{\rm crit}$ the agreement between the exact and the RQRPA and SCQRPA results is not anymore so good and worsens when Ω is increased. In any event the SCQRPA is preferable to the RQRPA, both close to the critical point and beyond.

In all previous studies of the O(5) model [26, 27, 28] only the excitations in the $(N-1, Z+1)$ systems have been studied. But, the energies E^+ could be as interesting for examining the different approximations as are the former ones. Thus, we show them in figure 6. The first issue that attracts attention, when comparing with the results presented in figure 5, is that the discrepancies among different QRPA theories for E^+, in the cases (A) and (B), are much more significant than for E^-. The dissimilarities with the exact results are also more pronounced, and especially in the former example. We do not find out any immediate explanation for this outcome. Yet, one should not be seriously worried by the results obtained in the case (A), as in this example there is only one proton in the final state. On the other hand, in the case (C) all three QRPA approaches for E^+ are quite similar before the collapse of the QRPA, and reproduce reasonable well the exact results. It should also be pointed out that only the SCQRPA is capable to account for the minimum of the exact E^+.

5.3 Double beta decay matrix elements

The matrix element for the $\beta\beta_{2\nu}$ decay can be cast in the form:

$$\mathcal{M}_{2\nu} = 2 \sum_l \frac{\langle \bar{0} | T_- | 0_l \rangle \langle 0_l | T_- | 0 \rangle}{E_l^- + \bar{E}_l^+}, \tag{5.8}$$

where $|0\rangle$ and $|\bar{0}\rangle$ stand, respectively, for the ground states in the initial (N, Z) and final $(N-2, Z+2)$ nuclei, with energies E_0 and $\bar{E}_0$, and

$$E_l^- = E_l - E_0; \qquad \bar{E}_l^+ = E_l - \bar{E}_0, \tag{5.9}$$

are the corresponding excitation energies from the ground states to the virtual states $|0_l\rangle$.

As said before, the matrix elements $\langle 0|T_-|0_l\rangle$ obey the sum rule (5.4) with $S^\pm$ given by (5.2). On the other hand, the matrix elements $\langle 0_l|T_-|\bar{0}\rangle$ fulfill the relation:

$$\bar{S}^- - \bar{S}^+ = N - Z - 4, \tag{5.10}$$

where

$$\bar{S}^\pm = \sum_l |\langle 0_l|T_\pm|\bar{0}\rangle|^2. \tag{5.11}$$

Within the QRPA calculations performed here there is only one intermediate state. Thus

$$\mathcal{M}_{2\nu} = 2\frac{\langle\bar{0}|T_-|0_{int}\rangle\langle 0_{int}|T_-|0\rangle}{E^- + \bar{E}^+}, \tag{5.12}$$

where, in the approximate calculations, $\langle 0_{int}|T_-|0\rangle$ and E^- are given, respectively, by eqs. (5.3) and (5.5),

$$\bar{E}^+ = \bar{\omega} - \bar{\lambda}_p + \bar{\lambda}_n, \tag{5.13}$$

and

$$\langle\bar{0}|T_-|0_{int}\rangle = \sqrt{2\Omega\bar{D}_{pn}}(\bar{\mathcal{Y}}\bar{u}_p\bar{v}_n + \bar{\mathcal{X}}\bar{v}_p\bar{u}_n). \tag{5.14}$$

Here the barred quantities correspond to the final nucleus.

Before proceeding, let us note that we have found numerically that, within the exact calculation, only one intermediate state contributes significantly. This state is the isobaric analog state (IAS) of the state $|0\rangle$ in the isospin symmetry limit, *i.e.* when $s = 1$ and $F_{pn} = 0$. Thus, the eq. (5.12) can also be used in the exact calculations for all practical purposes. Moreover, we can write

$$|\mathcal{M}_{2\nu}| = 2\frac{\sqrt{S^-\bar{S}^+}}{E^- + \bar{E}^+}. \tag{5.15}$$

Therefore, in the O(5) model the matrix element for the $\beta\beta_{2\nu}$ decay only depends on S^-, E^-, $\bar{S}^+$ and $\bar{E}^+$. The results for the first two quantities have been discussed already, and the results for the remaining two are shown in figures 7 and 8. It should be specified that the QRPA equations for the final systems collapse at significantly smaller values of s, than for the initial ones. Namely, now $s_{\rm crit} = 1.17,\ 1.13$ and 1.04, for the cases (A), (B) and (C), respectively. Nevertheless, from the comparison of figures 2 and 7 it can easily be seen that $\bar{S}^+$ is quite similar to S^+. In the same way from figures 6 and 8 one realizes

that $\bar{E}^+$ and E^+ are alike to each other, although the discrepancies between the exact and the approximates results are somewhat smaller for the former. Finally, in figure 9 are presented the results for the moments $\mathcal{M}_{2\nu}$. Note that the sign of $\mathcal{M}_{2\nu}$ is not physically relevant. Only for esthetic reasons we will choose it to be positive before becoming zero and negative afterwards. From the comparison with the exact calculations one can infer that the all three approximate calculations account for the general behavior of $\mathcal{M}_{2\nu}$, in the sense that they all cross the axis s close to the point where the exact calculations do it. They, however overestimate its magnitude, for $s \leq s_{\rm crit}$. From eq. (5.15) and the inspection of figures 2, 5, 7 and 8 it is easy to discover that the main reason for the discrepancies with the exact results comes from the differences in $\bar{E}^+$. Close to the critical value of s and beyond the SCQRPA is definitively "better" than RQRPA. Yet, in the case (C) all three approximations gives quite satisfactory results.

6 Summarizing Discussion

In this work we again took up the presently very debated issue of extending the quasiparticle RPA in the description of the charge-exchange transitions, to account more accurately for the physics around the transition point, which seems to be placed precisely in the region where the particle-particle force takes its physical value. An adequate tool to go beyond the QRPA approach seems to be its self-consistent extension, *i.e.* the SCQRPA. This approach properly includes ground state correlations and the Pauli principle, and has recently shown interesting results for the pairing problem [24] as well as for other physical systems [22, 29]. Because of the complexity of the problem it seems appropriate that this method be first tested on exactly solvable model cases before large scale realistic calculations are performed.

It is considered [26, 27, 28] that the schematic O(5) model is the appropriate testing ground for the double beta decay, and that it mimics rather accurately what is happening in realistic model calculations. But, one should be cautioned in doing such an extrapolation. First, the O(5) model is only able to describe the F transitions that play a marginal role in the $\beta\beta$ decay, in comparison with the GT transitions. Second, the F processes are related with the isospin conservation that is a good symmetry in nuclear physics, while the GT processes are associated with the Wigner SU(4) symmetry [36, 37], which is badly broken by the mean field and only partially restored by the residual interaction. Third, in realistic calculations the QRPA collapses in the physical region for the particle-particle strength only in the GT case [6, 8].

To study specifically the GT transitions within a schematic model one has to resort to the more general SO(8) model [4, 38] where the $T = 0$ and $T = 1$ pairings coexist. Recently [21] it was shown that the RQRPA, when applied to this model, fails in re-

producing the beta and double beta decay properties beyond the phase transition point (SU(4) limit). This failure was attributed to both: i) the need of changing the BCS basis, due to the transition to a $T = 0$ pairing region, and ii) the violation of the Ikeda sum rule. One might hope that the SCQRPA would solve the problem. Yet, although SO(8) is only slightly more complicated than O(5), we do not know how to build up explicitly the correlated ground state for the former, and, as was shown in the present work, this is the key point in solving the SCQRPA equation. The determination of the correlated ground state is also the main difficulty in applying the SCQRPA to real nuclei. Therefore, from the practical point of view, one of the most important tasks to solve the SCQRPA equations is the development of physically sound and numerically accurate approximate methods, which do not require the explicit construction of the correlated ground state. Schematic models such as the O(5) and SO(8) are perfect testing grounds for this trial.

In the present paper we have studied in detail the properties of the QRPA, RQRPA and SCQRPA equations in the O(5) model. Compared with the previous work [28], we have made several improvements, namely: i) we have solved exactly the SCQRPA equations of the O(5) model by evaluating everything exactly with the QRPA vacuum state (3.3.2) ii) we correctly took into account the coupling of the mean-field to the excitations. We also have examined carefully the RRPA limit of the model. The importance of all these features is clearly born out by the results. The energies, transition strengths, and double beta decay matrix elements are accounted for reasonably well. It clearly turns out that the SCQRPA approach performs better than either standard QRPA or renormalized QRPA for the description of these observables. Only in the case (C) the QRPA collapses in the physical region for the parameter s ($s \cong 1$). Thus, the samples (A) and (B) mimic better the F double beta decays and the example (C) the GT ones.

Before the collapse of the QRPA all three approaches reproduce well the exact results. Near the transition point only the SCQRPA values are close to the exact ones. Beyond that point both the SCQRPA and RQRPA yield different results from the exact ones, but the former are always qualitatively superior. It can be suspected that this situation even prevails to some extent in the realistic situation. On the other hand, it is very well known [35] that after the point where standard RPA breaks down, one has to change the single particle basis. In the present situation, where the proton-neutron correlations become "soft", this implies that after the phase transition point one has to consider the quasi-particles as mixed proton-neutron states.

Elaborating a SCQRPA in this new basis should allow to go in a smooth way through the phase transition (as happens in other models [22]), and we expect that then the SCQRPA will reproduce the exact results not only before but also beyond the transition point. In the same line one then also should include proton-proton and neutron-neutron pair excitation terms in the QRPA operator using the generalized Bogoljubov technique. Unfortunately such a generalization of the theory is a nontrivial task even in the O(5)

model and demands considerable formal developments. We intend to elaborate on this in the future.

A novel feature that has emerged from this work is that in the RRPA limit not only the proton-neutron components of the interaction are renormalized, as was well known previously [14], but also the quasi-particle energies. To get this effect it is essential to consider the coupling of the excitation to the mean-field, embodied in the generalized BCS equations [22].

One further technical aspect that deserves mentioning here concerns the inclusion of so-called anomalous terms $a^\dagger a$ in the excitation operator, as this was proposed in [23, 24]. Inclusion of such terms is very important to fulfill both the energy weighted and energy non weighted sum rules, as this was shown in [23] for the case of homogeneous infinite nuclear matter. A straightforward calculation also shows that such terms are essential to fulfill the sum rule for F and GT transitions in realistic finite nuclei. Inclusion of these terms is also relevant for the correct treatment of symmetries and the appearance of the Golstone mode [23]. But, contrary to the infinite matter case, the consistent inclusion of such terms in group theoretical models, like the one treated here, is a delicate problem and demands further studies.

A particular feature of our model relates to the fact that the QRPA ground state wave function could be constructed explicitly and therefore the system of equations could be closed without further approximations. In the realistic situation it is certainly far from obvious to obtain the ground state wave function and approximations must be used to close the system. In this respect efficient methods have been developed in the past, like e.g. the number operator method [30]. On the other hand, it has been shown that the system of equations can be closed without knowing the wave function in coupling the SCRPA to the one body Green's function [23]. This way was considered in eq. (16) of ref. [24]. In the future the elaboration of a more consistent single particle equation coupled to SCRPA will be a key point in closing the equations also for realistic situations.

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Appendix A: Expectation values within the SCQRPA

Except for the operators $\mathcal{A}^\dagger_{pp}\mathcal{A}^\dagger_{nn}$, $\mathcal{A}^\dagger_{pp}\mathcal{A}_{pp}$ and $\mathcal{N}_{np}\mathcal{N}_{pn}$, the calculation of the expectation values can be easily done using the completeness properties of the SCQRPA exchange mode and the properties of the basis states. For these three operators we had to use the generator state technique from ref. [39].

Using the notation,

$$\langle\mathcal{N}\rangle \equiv \langle 0|\mathcal{N}_{pp}|0\rangle = \langle 0|\mathcal{N}_{nn}|0\rangle, \tag{A.1}$$

and the ground state given by (3.3.2) we get

$$\langle\mathcal{N}\rangle = 2N_0 \sum_{l=0}^{\Omega} l\alpha_l^2 z^{2l}\langle 2l|2l\rangle, \tag{A.2}$$

$$\langle 0|\mathcal{N}^2_{pp}|0\rangle = \langle 0|\mathcal{N}^2_{nn}|0\rangle = \langle 0|\mathcal{N}_{nn}\mathcal{N}_{pp}|0\rangle = 4N_0 \sum_{l=0}^{\Omega} l^2\alpha_l^2 z^{2l}\langle 2l|2l\rangle, \tag{A.3}$$

$$\langle 0|\mathcal{A}^\dagger_{pp}\mathcal{A}^\dagger_{nn}|0\rangle = -\frac{4N_0}{2\Omega-1}\sum_{l=1}^{\Omega} l(2\Omega-2l+1)\alpha_l^2 z^{2l-1}\langle 2l|2l\rangle, \tag{A.4}$$

$$\langle 0|\mathcal{A}^\dagger_{pp}\mathcal{A}_{pp}|0\rangle = \langle 0|\mathcal{A}^\dagger_{nn}\mathcal{A}_{nn}|0\rangle = \frac{4N_0}{2\Omega-1}\sum_{l=1}^{\Omega} l(2l-1)\alpha_l^2 z^{2l}\langle 2l|2l\rangle, \tag{A.5}$$

$$\langle 0|\mathcal{A}^\dagger_{pn}\mathcal{A}_{pn}|0\rangle = \frac{2z^2}{1-z^2}(\Omega - \langle\mathcal{N}\rangle), \tag{A.6}$$

$$\langle 0|\mathcal{A}_{pn}\mathcal{A}_{pn}|0\rangle = \langle 0|\mathcal{A}^\dagger_{pn}\mathcal{A}^\dagger_{pn}|0\rangle = \frac{2z}{1-z^2}(\Omega - \langle\mathcal{N}\rangle), \tag{A.7}$$

and

$$\langle 0|\mathcal{N}_{np}\mathcal{N}_{pn}|0\rangle = \langle 0|\mathcal{N}_{pn}\mathcal{N}_{np}|0\rangle = 2\langle\mathcal{N}\rangle - \langle 0|\mathcal{A}^\dagger_{pp}\mathcal{A}_{pp}|0\rangle. \tag{A.8}$$

Appendix B: RQRPA limit for the expectation values

With $\langle\mathcal{N}\rangle$ defined in (A.1) we obtain

$$\langle 0|\mathcal{N}_{pp}^2|0\rangle = \langle 0|\mathcal{N}_{nn}^2|0\rangle = \langle\mathcal{N}\rangle^2 + \langle\mathcal{N}\rangle\left(1 - \frac{\langle\mathcal{N}\rangle}{2\Omega}\right), \tag{B.1}$$

$$\langle 0|\mathcal{N}_{nn}\mathcal{N}_{pp}|0\rangle = \langle\mathcal{N}\rangle^2, \tag{B.2}$$

$$\langle 0|\mathcal{A}_{pp}^\dagger\mathcal{A}_{nn}^\dagger|0\rangle = \langle 0|\mathcal{A}_{pn}^\dagger\mathcal{A}_{pn}^\dagger|0\rangle = \langle 0|\mathcal{A}_{pn}\mathcal{A}_{pn}|0\rangle = 0, \tag{B.3}$$

$$\langle 0|\mathcal{A}_{pp}^\dagger\mathcal{A}_{pp}|0\rangle = \langle 0|\mathcal{A}_{nn}^\dagger\mathcal{A}_{nn}|0\rangle = 2\langle 0|\mathcal{A}_{pn}^\dagger\mathcal{A}_{pn}|0\rangle = \frac{\langle\mathcal{N}\rangle^2}{\Omega}, \tag{B.4}$$

and

$$\langle 0|\mathcal{N}_{np}\mathcal{N}_{pn}|0\rangle = \langle 0|\mathcal{N}_{pn}\mathcal{N}_{np}|0\rangle = \langle\mathcal{N}\rangle - \frac{\langle\mathcal{N}\rangle^2}{2\Omega} = \langle\mathcal{N}\rangle\left(1 - \frac{\langle\mathcal{N}\rangle}{2\Omega}\right). \tag{B.5}$$

The following relations are also useful:

$$\langle 0|\left(2\Omega - \mathcal{N}\right)^2|0\rangle = 4\Omega^2 + \frac{\langle\mathcal{N}\rangle}{\Omega}(4\Omega - 1)(\langle\mathcal{N}\rangle - 2\Omega), \tag{B.6}$$

$$\langle 0|\left(\Omega - \mathcal{N}_{pp}\right)^2|0\rangle = \Omega^2 + \frac{\langle\mathcal{N}\rangle}{2\Omega}(2\Omega - 1)(\langle\mathcal{N}\rangle - 2\Omega), \tag{B.7}$$

$$\langle 0|2\left(\Omega - \mathcal{N}_{pp}\right)^2 - \mathcal{A}_{pp}^\dagger\mathcal{A}_{pp}|0\rangle \;=\; 2\left(\Omega^2 + \langle\mathcal{N}\rangle(1 - 2\Omega) + \frac{\langle\mathcal{N}\rangle^2}{\Omega}(\Omega - 1)\right). \tag{B.8}$$

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Figure Captions

Figure 1: Frequencies ω for the (N, Z) systems for the cases (A), (B) and (C), as a function of the particle-particle coupling constant s.

Figure 2: Transition strengths S^- for the (N, Z) systems for the cases (A), (B) and (C), as a function of the particle-particle coupling constant s.

Figure 3: The ratio $z = \mathcal{Y}/\mathcal{X}$ (upper panel), expectation value of the number of quasiparticles in the ground state $\langle\mathcal{N}\rangle = \langle 0|\mathcal{N}_p|0\rangle = \langle 0|\mathcal{N}_n|0\rangle$ (middle panel), and the occupation probability for neutrons v_n^2 (lower panel). Within the QRPA, $\langle\mathcal{N}\rangle = 2\mathcal{Y}^2$.

Figure 4: Excitation energies E^- for the case (B), as a function of the particle-particle coupling constant s. The approximate results were obtained with the prescriptions (5.5) (upper panel) and (5.7) (lower panel).

Figure 5: Excitation energies E^- for the cases (A), (B) and (C), as a function of the particle-particle coupling constant s. The approximate results were obtained from eq. (5.5).

Figure 6: Excitation energies E^+ for the cases (A), (B) and (C), as a function of the particle-particle coupling constant s. The approximate results were obtained from eq. (5.5).

Figure 7: Transition strengths $\bar{S}^+$ for the $(N-2, Z+2)$ systems for the cases (A), (B) and (C), as a function of the particle-particle coupling constant s.

Figure 8: Excitation energies $\bar{E}^+$ for the cases (A), (B) and (C), as a function of the particle-particle coupling constant s. The approximate results were obtained from eq. (5.13).

Figure 9: Matrix element $\mathcal{M}_{2\nu}$ as a function of the particle-particle coupling constant s.

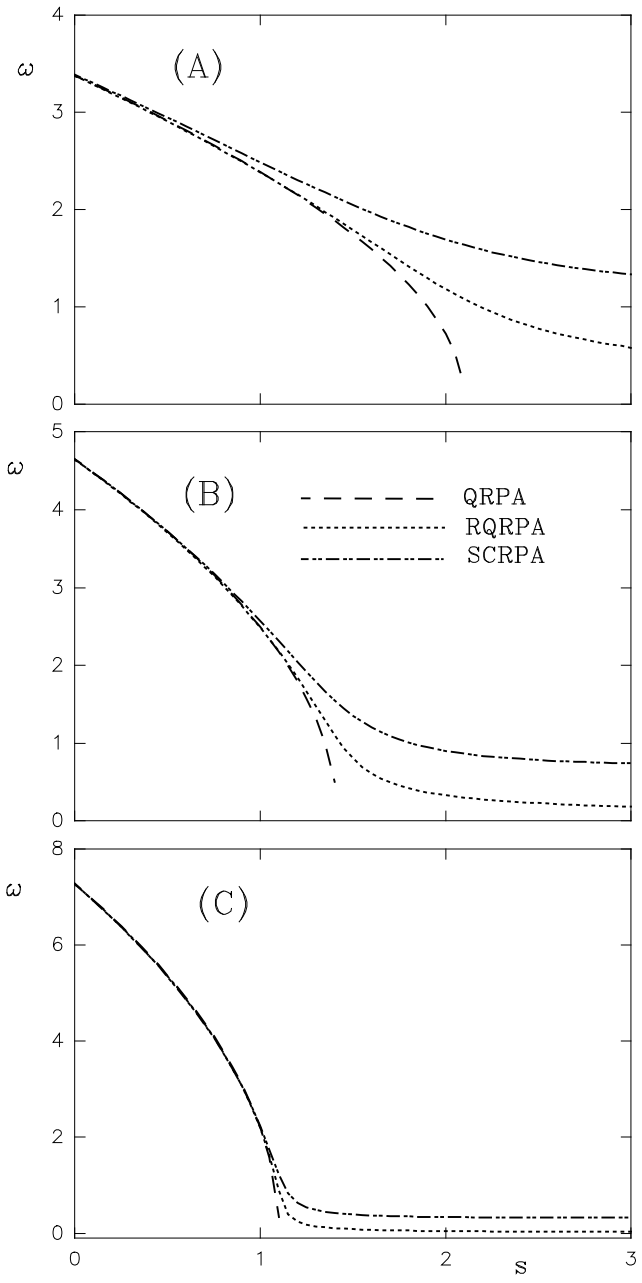


Fig. 1

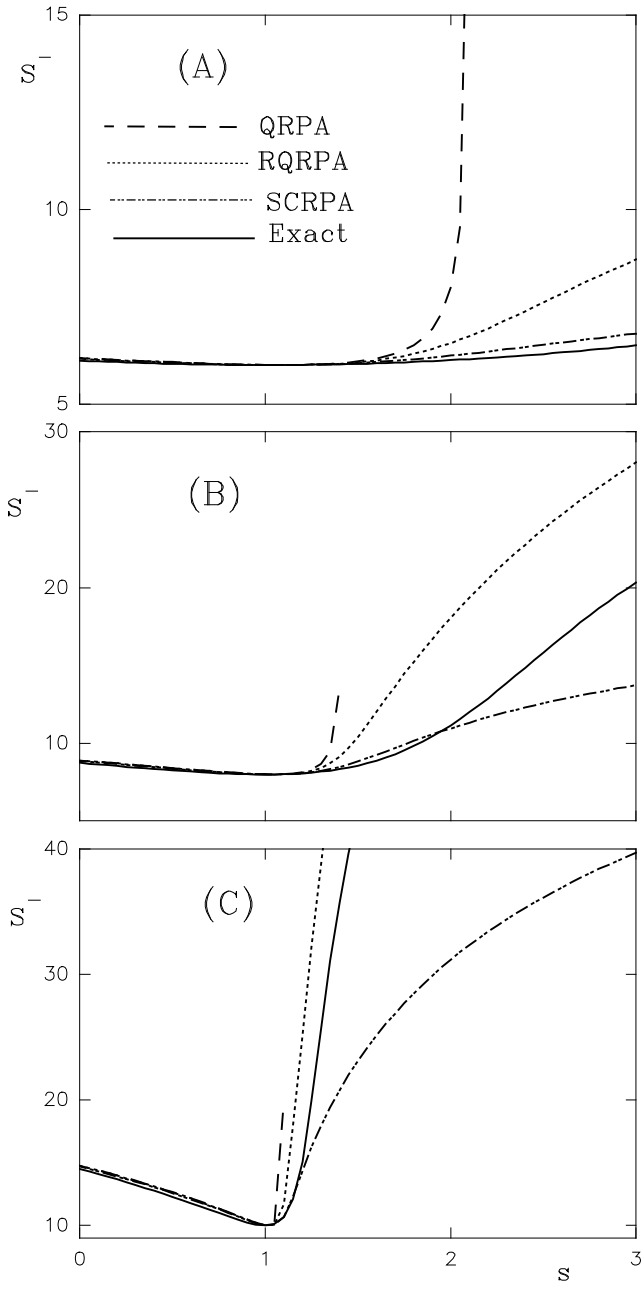
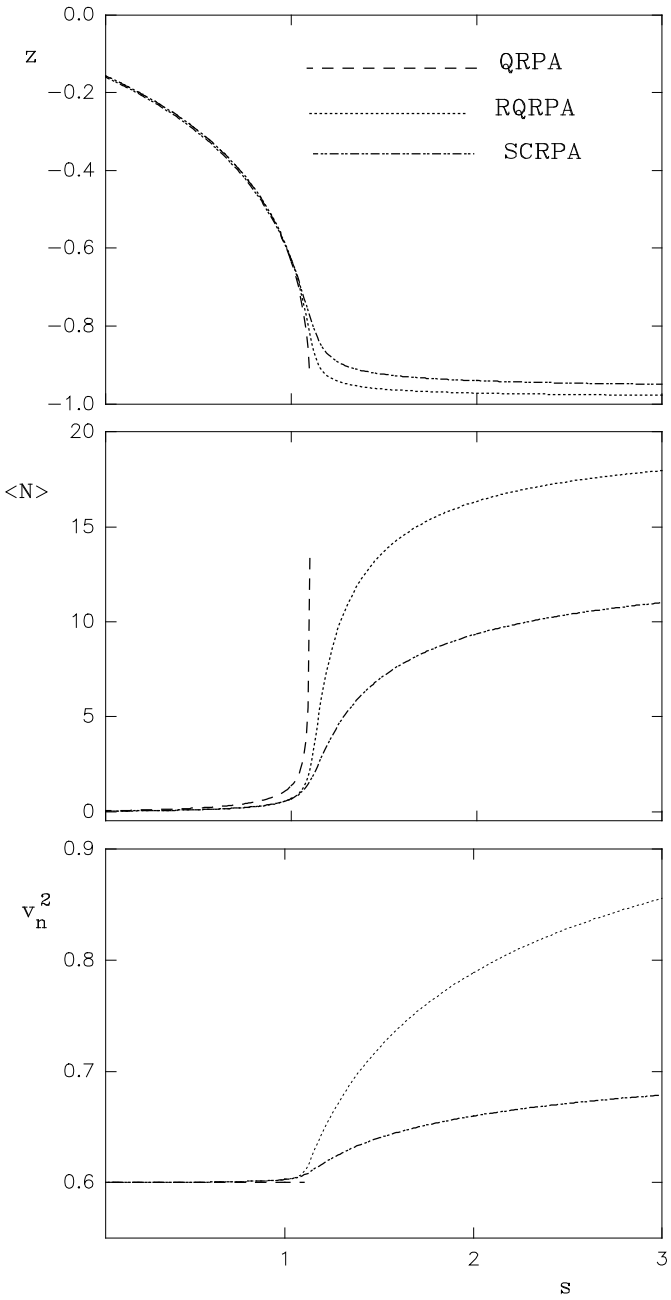


Fig.2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v_n^2
0.8
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1
2
3
s

Fig. 3

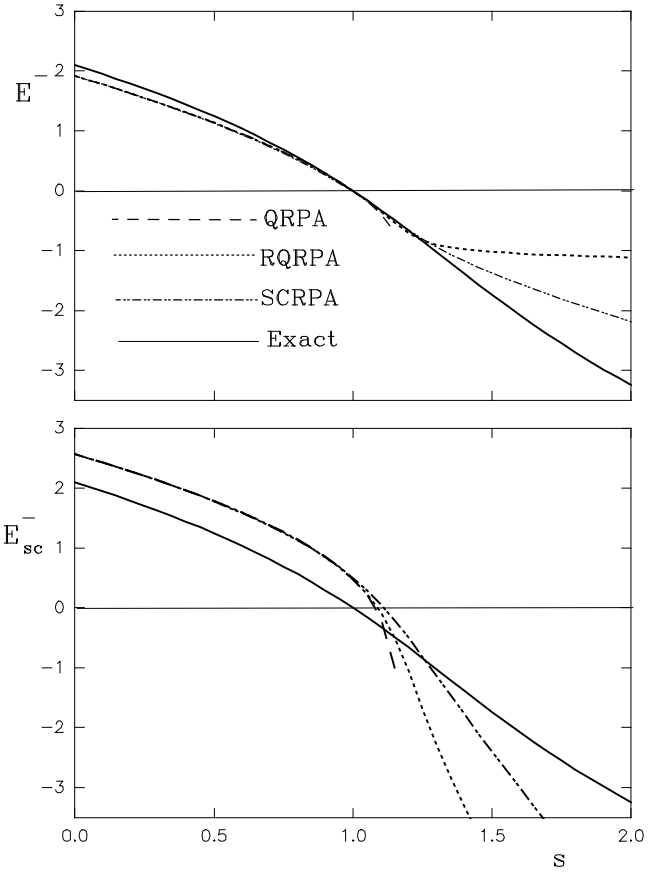

Fig. 4

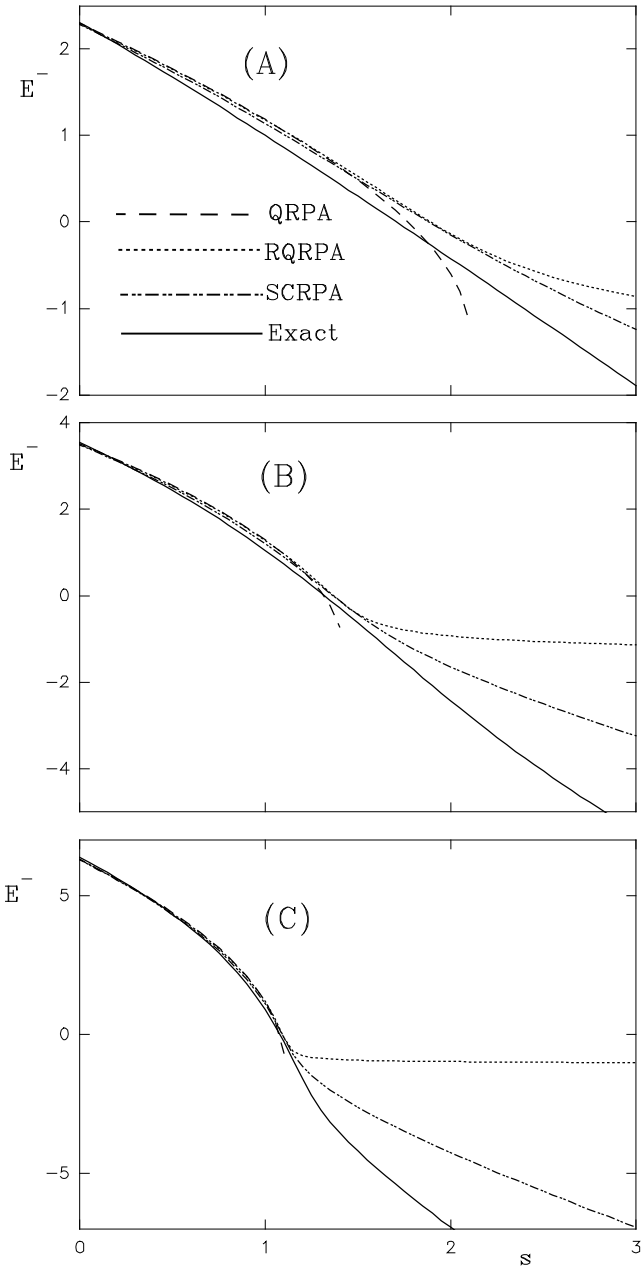

Fig. 5

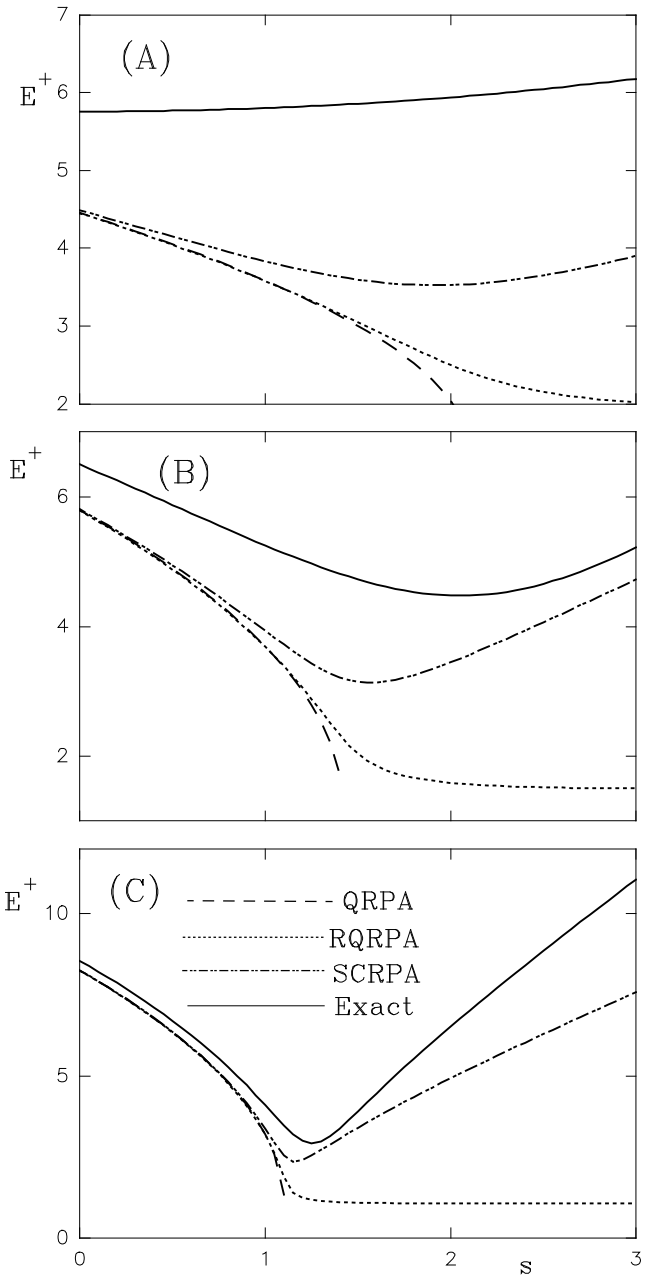

Fig.6

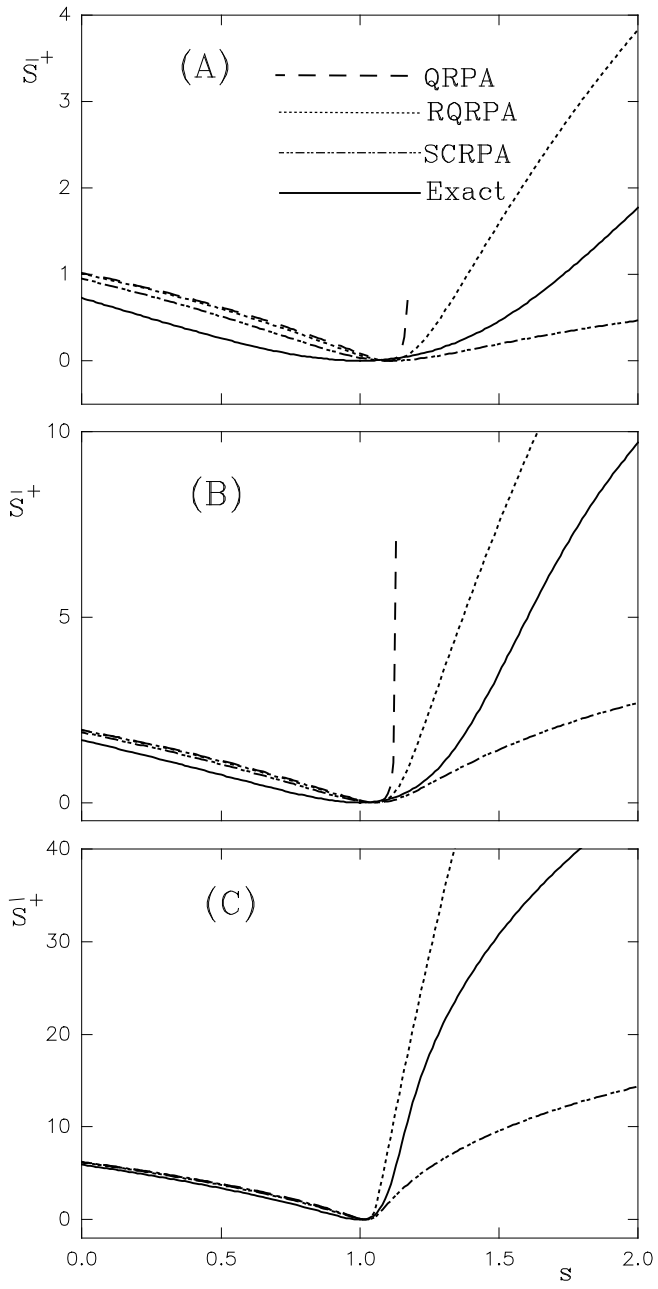

Fig.7

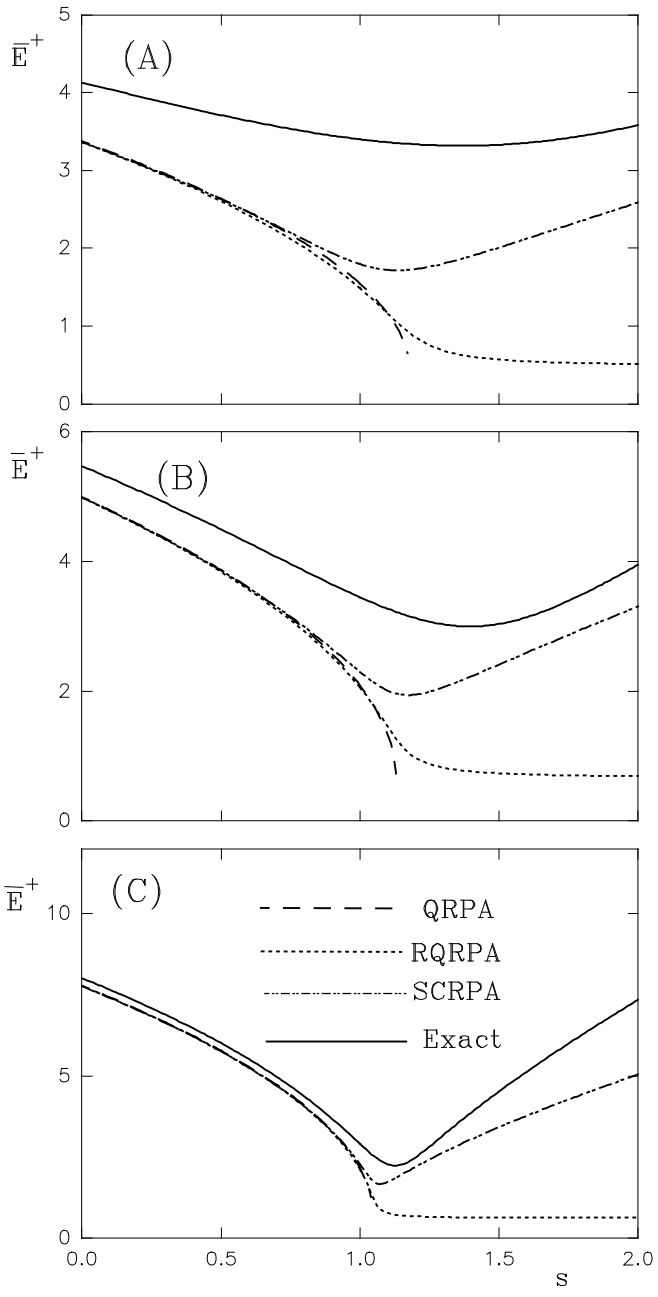

Fig.8

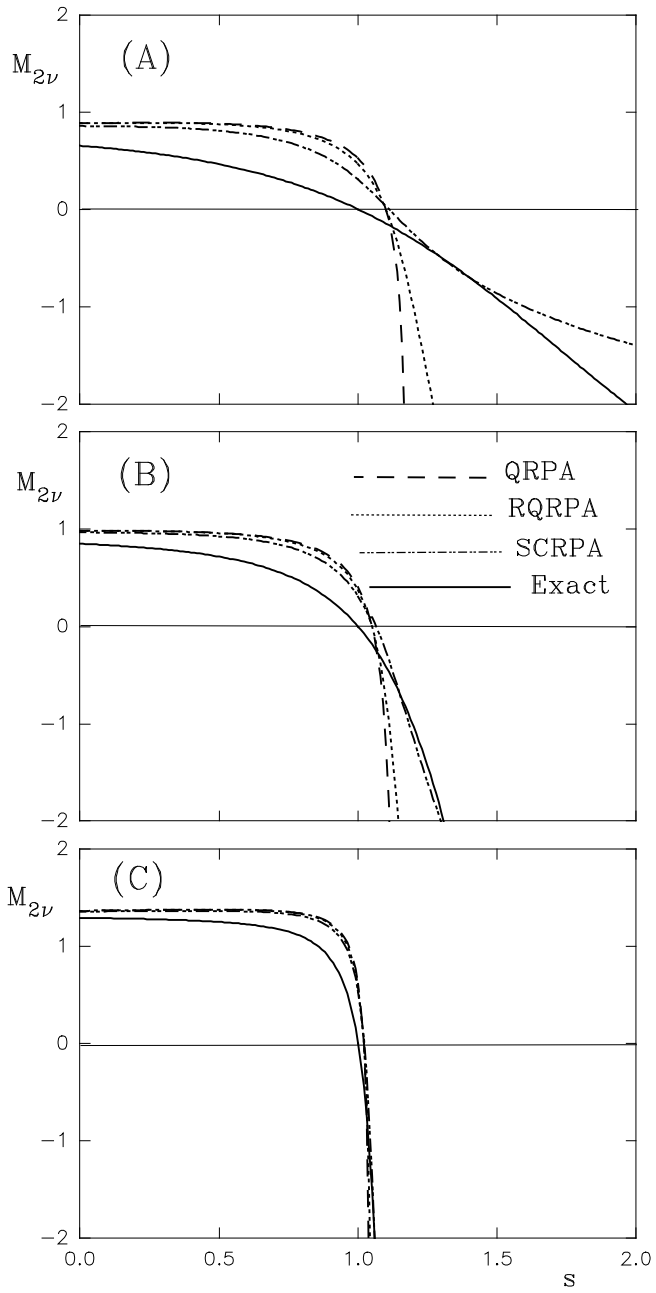

Fig.9